Relative Controllability of Switched Multi-agent Systems Subject to Communication Delay

Xiaobin Yu, Hongwei Shi, Yuanchao Si, and Ji-Feng Zhang, *Fellow, IEEE*

***Abstract*—This paper is concerned with the relative controllability of leader-follower multi-agent systems (MASs) with communication delay and switching topologies, where the roles and quantities of leaders (resp. followers) dynamically depends on topology variations. By compressing the MASs with communication delay and switching topologies into a switched system with state delay, a new method to construct the solution of the switched system is proposed. Then, a delay-dependent nested subspace is given, based on which, a relative controllability criterion associated with the controllable subspace is established, and a delay-dependent Kalman-type block matrix and an algebraic criterion for relative controllability of MASs are obtained. In addition, the critical terminal time for achieving relative controllability and the minimum energy required for relative controllability are also given.**

***Index Terms*—Communication delay, minimum energy, multi-agent systems, relative controllability, switching topologies.**

I. Introduction

Since Kalman introduced the concept of controllability for dynamic systems in 1960s, it has became a fundamental property of systems. To control a dynamic system, we first need to determine whether it is controllable. This intrinsic motivation underlies research on controllability of a system. Relative controllability is a significant concept that extends classical controllability and is specifically proposed for time-delay systems. Research has demonstrated that relative controllability in time-delay systems exhibits distinctive properties [1]–[3]. For instance, achieving relative controllability requires that the terminal time exceeds a delay-dependent constant and the Kalman-type matrix, which is constructed from system matrix pairs and time-delay, is of full row rank [1].

Multi-agent systems (MASs) consist of multiple agents with local perception, communication, and computing capabilities, and are widely applied in artificial intelligence [4], communications [5], biology [6], and other fields. The controllability of MASs has been a fundamental research problem since Tanner's seminal discovery of the relationship between controllability properties and topological structure [7]. Furthermore, controllability is a prerequisite condition for solving the consensus problem in MASs with packet loss [8]. Owing to its significant theoretical importance and practical applications, the controllability of MASs has become an important research focus [9]–[11]. To name a few, Qu et al. investigate the controllability of MASs under equitable partitions [9] and Guan et al. study controllability from the algebraic and graphical perspectives [11]. Other important research directions in MASs controllability include structural controllability [11], target controllability [12] and group controllability [13], etc.

The controllability of MASs with switching topologies has attracted considerable attention as an important research topic [14]–[16], since it has been proved that topology switching can fundamentally alter their controllability properties [14], [16]. Si et al. have ever discussed relative controllability of the MASs with input delay and switching topologies from space perspective [15]. Nevertheless, the critical terminal time required for switched MASs with delay to attain relative controllability remains an open problem. This paper aims to solve this problem. Specifically, we further investigate the relative controllability of MASs with communication delay and switching topologies. Compared with the existing literature, a key characteristic of the MASs we consider lies in the fact that the rules and quantities of leaders (resp. followers) vary with topology switching. This model more closely reflects both the flexibility and practical applications of MASs. However, it also results in variations in the dimensions of the control exerted on followers in each topology. Furthermore, although the protocol we adopt is widely used in multi-agent consensus research, achieving the corresponding relative controllability for this protocol remains an open challenge.

The primary contributions of this work can be summarized as follows:

1. By compressing the MASs with communication delay and switching topologies into a switched system with state delay, a new method to construct the solution of the switched system is proposed. Particularly, the switched system is first decomposed into a zero-forcing switched subsystem with nonzero initial function and a nonzero-forcing switched subsystem with zero initial function. It is shown that solution of the switched system is the sum of the zero-forcing and nonzero-forcing switched subsystem solutions. Then, by using the matrix delay exponential, several new nested functions are given to derive the explicit solution of the nonzero-forcing switched system.

2. A novel delay-dependent nested subspace is constructed, which is shown to be equal to the subspace spanned by the

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Xiaobin Yu, Hongwei Shi and Yuanchao Si are with the School of Mathematics and Statistics, Guizhou University, Guiyang 550025, Guizhou, China (email:xbyu2024@126.com; shihongwei2024@126.com; ycsi@gzu.edu.cn).

Ji-Feng Zhang is with the School of Automation and Electrical Engineering, Zhongyuan University of Technology, Zhengzhou 450007, Henan, China, and also with the State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China. (e-mail: jif@iss.ac.cn).

columns of Gramian-type matrix. Based on this, a relative controllability criterion associated with the controllable subspace is established. Moreover, the critical terminal time required to achieve relative controllability of the MASs is given.

3. A novel delay-dependent Kalman-type block matrix and an algebraic criterion for relative controllability of MASs are derived. Furthermore, how the critical terminal time influences the rank property of the Kalman-type matrix is characterized.

4. The external control effort required for state-to-goal transitions is quantified by optimizing a functional index, and the minimum control energy needed to achieve the desired objectives is presented.

The remainder of this paper is organized as follows. Section II introduces fundamental graph theory concepts and some important notations. Section III formulates the MASs with communication delay and switching topologies as a switched delay system. Section IV investigates the relative controllability of the proposed switched system. The minimum energy achieving the relative controllability is analyzed in Section V. Finally, a numerical example is presented in Section VI.

II. Preliminaries

An undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a vertex set $\mathcal{V} = \{1, \cdots, n\}$ and an edge set $\mathcal{E} \subseteq \{(j,k) : j,k \in \mathcal{V}\}$, where edge (j,k) signifies a connection between j and k. If $j,k \in \mathcal{V}$ and $(j,k) \in \mathcal{E}$, we say that k is a neighbor of j, denoted by $k \bowtie j$. The neighbor set of vertex j is defined as $\mathcal{N}_j \triangleq \{k | k \bowtie j,\ k \in \mathcal{V}\}$ and its cardinality is denoted by $|\mathcal{N}_j|$. A weighted graph $\mathcal{G} = (W, \mathcal{V}, \mathcal{E})$ is a graph equipped with weight matrix $W = [w_{jk}] \in \boldsymbol{\mathcal{R}}^{n\times n}$, elements of which are defined as follows: If $k \bowtie j$, then $w_{kj} > 0$; otherwise, $w_{kj} = 0$, $j,k \in \{1, \cdots, n\}$. The weight degree matrix of $\mathcal{G}$ is a diagonal matrix denoted by $D = \mathrm{Diag}\{d_1, \cdots, d_n\}$, where $d_i = \sum_{j=1}^{n} w_{ij}$. The Laplacian matrix of $\mathcal{G}$ is defined as $L = D - W$.

In what follows, we denote by $\boldsymbol{\mathcal{R}}$ the set of real numbers, $\boldsymbol{\mathcal{N}}$ the set of nonnegative integers and $\boldsymbol{\mathcal{N}}^+$ the set of positive integers. Define $\overline{m} = \{1, \cdots, m\}$ for an integer $m > 0$ and $\overline{m} = \emptyset$ for $m \leq 0$. For a real number $\alpha \in \boldsymbol{\mathcal{R}}$, let $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α. Given a matrix $B \in \boldsymbol{\mathcal{R}}^{n\times m}$, its column space and null space are respectively defined as: $\mathbf{Col}B = \{\xi | \xi = B\eta,\ \eta \in \boldsymbol{\mathcal{R}}^m\}$ and $\mathbf{Nul}B = \{\eta | B\eta = 0, \eta \in \boldsymbol{\mathcal{R}}^m\}$. Given two subspaces $\boldsymbol{\mathcal{S}}_1$ and $\boldsymbol{\mathcal{S}}_2$, define $\boldsymbol{\mathcal{S}}_1 + \boldsymbol{\mathcal{S}}_2 = \{x + y | x \in \boldsymbol{\mathcal{S}}_1, y \in \boldsymbol{\mathcal{S}}_2\}$. For a matrix B and a subspace $\boldsymbol{\mathcal{S}}$, denote $B\boldsymbol{\mathcal{S}} = \{Bx | x \in \boldsymbol{\mathcal{S}}\}$ and $\boldsymbol{\mathcal{S}}^\perp$ the orthogonal complement of $\boldsymbol{\mathcal{S}}$. Denoted by $\mathbf{1}_n = [1, \cdots, 1]^T$. Let Θ and I denote the zero matrix and unit matrix of appropriate dimensions, respectively. Given a matrix $A \in \boldsymbol{\mathcal{R}}^{n\times n}$ and a constant $\tau > 0$, the matrix delay exponential e_τ^{At} is defined as a piecewise matrix polynomial [17]:

$$e_\tau^{At} = \begin{cases} \Theta, & -\infty < t < -\tau, \\ I, & -\tau \leqslant t < 0, \\ \sum\limits_{j=0}^{k} A^j \frac{(t-(j-1)\tau)^j}{j!}, & (k-1)\tau \leqslant t < k\tau, k \in \boldsymbol{\mathcal{N}}^+. \end{cases} \tag{1}$$

This function satisfies $\frac{de_\tau^{At}}{dt} = Ae_\tau^{A(t-\tau)}$ for $t > 0$, with initial condition $e_\tau^{At} = I$ for $t \in [-\tau, 0]$. More properties of the matrix delay exponential can be found in [1], [2].

III. Formulation

Suppose that the leader-follower structured MAS is composed of n agents labeled from 1 to n and the communication topology among agents is switching. Assume that there are r switching topologies in the MAS. Each topology of the MAS is abstracted as an undirected graph $\mathcal{G}^{[i]} = (W^{[i]}, \mathcal{V}, \mathcal{E}^{[i]})$ with $W^{[i]} = [w_{jl}(i)] \in \boldsymbol{\mathcal{R}}^{n\times n}$, $\mathcal{V}$ and $\mathcal{E}^{[i]}$ denoting the weight matrix, the vertex set and the edge set, respectively, $i \in \overline{r}$. The quantities and roles of leaders (resp. followers) are varying with the switching signal, and the edges and corresponding weights also change with time. For any $i \in \overline{r}$, the leaders are assigned to the set $\mathcal{C}_L(i)$ and the followers are to the set $\mathcal{C}_F(i)$. Clearly, $\mathcal{C}_L(i) \cup \mathcal{C}_F(i) = \mathcal{V}$ and $|\mathcal{C}_L(i)| + |\mathcal{C}_F(i)| = n$, $i \in \overline{r}$. Due to limitations in bandwidth or signal interference, interactions among agents are always subjected to communication delays. Accordingly, we assume that inter-agent communication is affected by time delay. The agents update their states according to the following rules: For $j \in \mathcal{C}_F(\sigma(t))$,

$$\dot{x}^{[j]}(t) = \sum_{k \in \mathcal{N}_j(\sigma(t))} w_{jk}(\sigma(t)) \left(x^{[k]}(t-\tau) - x^{[j]}(t-\tau)\right), \tag{2}$$

and for $j \in \mathcal{C}_L(\sigma(t))$,

$$\dot{x}^{[j]}(t) = \sum_{k \in \mathcal{N}_j(\sigma(t))} w_{jk}(\sigma(t)) \left(x^{[k]}(t-\tau) - x^{[j]}(t-\tau)\right) + u_{\sigma(t)}^{[j]}(t), \tag{3}$$

where $x^{[j]} \in \boldsymbol{\mathcal{R}}$ is the state of agent j, $\sigma : [0, \infty) \to \overline{r}$ is the switching signal that determines the active communication topology at time t, $\mathcal{N}_j(\sigma(t))$ is the neighbor set of agent j corresponding to the $\sigma(t)$-th topology, $w_{kj}(\sigma(t))$ is the weight of edge (j,k), $u_{\sigma(t)}^{[j]} \in \boldsymbol{\mathcal{R}}$ represents the external control input exerted on leader j, and $\tau > 0$ is the communication delay.

The protocol in (2)–(3) considers the influence of communication delay and integrates switching topologies to enhance the obstacle avoidance capability of the MAS. This ensures the controllability of entire MAS even when individual subsystems are uncontrollable. Although the protocol in (2)–(3) is widely used in consensus research [16], the corresponding controllability, as a fundamental problem, remains unsolved. To study the optimal control of (2)–(3) as in [18], we must first establish the controllability of the MAS. Since the quantities and roles of leaders (resp. followers) vary with topology switching, the conventional approach, which treats leaders as control inputs for followers and analyzes only the controllability of followers, faces significant challenges.

Let $x = (x^{[1]}, \cdots, x^{[n]})^T \in \boldsymbol{\mathcal{R}}^n$ and $u_{\sigma(t)} = (u_{\sigma(t)}^{[j_1]}, \cdots, u_{\sigma(t)}^{[j_{m(\sigma(t))}]})^T \in \boldsymbol{\mathcal{R}}^{m(\sigma(t))}$, where $j_1, \cdots, j_{m(\sigma(t))} \in \mathcal{C}_L(\sigma(t))$ and $m(\sigma(t)) = |\mathcal{C}_L(\sigma(t))|$. Denote by $B_{\sigma(t)}$ the input matrix associated with the $\sigma(t)$-th topology, whose columns are the standard basis vectors associated with the leader indices. Thus, the MAS (2)-(3) is further modeled as the following switched system with state delay:

$$\dot{x}(t) = -L_{\sigma(t)} x(t-\tau) + B_{\sigma(t)} u_{\sigma(t)}(t), \tag{4}$$

where $L_{\sigma(t)} = [l_{jk}(\sigma(t))] \in \boldsymbol{\mathcal{R}}^{n\times n}$ is the Laplacian matrix under the $\sigma(t)$-th topology. Notice that (4) consists of

r subsystems, each of which is abbreviated as a matrix pair $(-L_j, B_j)_\tau$, $j \in \overline{r}$. If $\sigma(t) = j$, the j-th subsystem $(-L_j, B_j)_\tau$ is triggered, $j \in \overline{r}$. If the MAS is not affected by delay, then (4) reduces to the classical switched linear system, each subsystem of which is represented by $(-L_j, B_j)$, $j \in \overline{r}$.

IV. Relative Controllability

To investigate the relative controllability of (4), we need to begin with its solution. To this end, we first introduce the concept of switching sequence with referring to [19].

Definition 1. *A set $\mathcal{S}$ of ordered pairs $(i_q, \kappa_q), q \in \overline{p}$ is called a switching sequence, where $\mathcal{S} \triangleq \{(i_1, \kappa_1), \cdots, (i_p, \kappa_p)\}$, p represents the length of $\mathcal{S}$, $i_q \in \overline{r}$ corresponds to the q-th subsystem on the interval $[t_{q-1}, t_q)$ and $\kappa_q = t_q - t_{q-1}$ represents the dwell time of the q-th subsystem, $q \in \overline{p}$.*

In what follows, we assume that the dwell time of each subsystem satisfies $\kappa_q = l_q \tau$, $l_q \in \boldsymbol{\mathcal{N}}^+$, $q \in \overline{p}$. For clarity in subsequent derivations, we denote the solution of the subsystem corresponding to $\sigma(t) = i_q$ by $x_q(t)$, $t \in [t_{q-1}, t_q)$, $q \in \overline{p}$. Note that when $\sigma(t) = i_p$, the running interval of the subsystem is $[t_{p-1}, t_p]$. For simplicity, we stipulate that $t_0 = 0$.

The state of the system with delay is closely related to the initial function. Thus, given an initial function $\varphi(t)$, $t \in [-\tau, 0)$ and a switching sequence $\mathcal{S} = \{(i_q, \kappa_q)\}_{q=1}^p$, we redefine (4) as follows:

$$\begin{cases} \dot{x}_q(t) = -L_{i_q} x_q(t-\tau) + B_{i_q} u_{i_q}(t), \ t \in [t_{q-1}, t_q), \\ x_q(t) = x_{q-1}(t), \qquad\qquad\qquad t \in [t_{q-1} - \tau, t_{q-1}), \end{cases} \tag{5}$$

where $x_0(t) = \varphi(t)$ for $t \in [-\tau, 0)$ and $q \in \overline{p}$. (For $q = p$, the time interval is $[t_{p-1}, t_p]$).

From (5), we know that the state of the q-th subsystem is identical to that of the $(q-1)$-th subsystem at $[t_{q-1} - \tau, t_{q-1})$. This implies that the switching of the MAS among topology graphs occurs without jumps. Besides, L_{i_q}, B_{i_q} and u_{i_q} all vary with switching function. Thus, (5) is an extension of the classical switched system.

A. Solution of the Switched System

Next, we consider the solution of (5). Given a switching sequence $\mathcal{S} = \{(i_q, \kappa_q)\}_{q=1}^p$ and an initial function $\varphi(t)$, $t \in [-\tau, 0)$, we can verify that iterative solution of (5), applying the matrix delay exponential (1), can be represented as $x_q(t) = \int_{t_{q-1}-\tau}^{t_{q-1}} e_\tau^{-L_{i_q}(t-\tau-s_q)} \dot{x}_{q-1}(s_q) ds_q + \int_{t_{q-1}}^{t} e_\tau^{-L_{i_q}(t-\tau-s_q)} B_{i_q} u_{i_q}(s_q) ds_q + e_\tau^{-L_{i_q}(t-t_{q-1})} x_{q-1}(t_{q-1} - \tau)$, where $x_0(t) = \varphi(t)$, $q \in \overline{p}$. Following [15], we further decompose $x_q(\cdot)$ as

$$x_q(t) = y_q(t) + z_q(t), \tag{6}$$

where $y_q(\cdot)$ and $z_q(\cdot)$, $q \in \overline{p}$, are respectively defined as follows: $y_q(t) = \int_{t_{q-1}-\tau}^{t_{q-1}} e_\tau^{-L_{i_q}(t-\tau-s_q)} \dot{y}_{q-1}(s_q) ds_q + e_\tau^{-L_{i_q}(t-t_{q-1})} y_{q-1}(t_{q-1} - \tau)$ with $y_0(t) = \varphi(t)$, $t \in [-\tau, 0)$ and $z_q(t) = \int_{t_{q-1}-\tau}^{t_{q-1}} e_\tau^{-L_{i_q}(t-\tau-s_q)} \dot{z}_{q-1}(s_q) ds_q + \int_{t_{q-1}}^{t} e_\tau^{-L_{i_q}(t-\tau-s_q)} B_{i_q} u_{i_q}(s_q) ds_q + e_\tau^{-L_{i_q}(t-t_{q-1})} z_{q-1}(t_{q-1} - \tau)$ with $z_0(t) = 0$, $t \in [-\tau, 0)$. It is not difficult to prove that $y_q(\cdot)$, $q \in \overline{p}$ is a solution of

$$\begin{cases} \dot{y}_q(t) = -L_{i_q} y_q(t-\tau), \ t \in [t_{q-1}, t_q), \\ y_q(t) = y_{q-1}(t), \ \ t \in [t_{q-1} - \tau, t_{q-1}), \end{cases} \tag{7}$$

with $y_0(t) = \varphi(t)$, $t \in [-\tau, 0)$. Similarly, we can verify that $z_q(\cdot)$, $q \in \overline{p}$ satisfies

$$\begin{cases} \dot{z}_q(t) = -L_{i_q} z_q(t-\tau) + B_{i_q} u_{i_q}(t), \ t \in [t_{q-1}, t_q), \\ z_q(t) = z_{q-1}(t), \qquad\qquad\qquad t \in [t_{q-1} - \tau, t_{q-1}), \end{cases} \tag{8}$$

where $z_0(t) = 0$, $t \in [-\tau, 0)$. Thus, the system (5) is further decomposed as (7) and (8), a zero-forcing system and a nonzero-forcing system. Note that $y_q(\cdot)$, $q \in \overline{p}$, is only related to initial function $\varphi(\cdot)$ and switching sequence $\mathcal{S}$. Thus, we claim that the relative controllability of (5) is mainly determined by (8). This renders us to investigate the structure of $z_q(\cdot)$, $q \in \overline{p}$.

For $q \in \overline{p} \setminus \{1\}$, construct functions as follows:

$$\psi_{q,j}(t,s) = \begin{cases} e_\tau^{-L_{i_q}(t-t_{q-1})} \left(\prod_{h=q-1}^{j+1} e_\tau^{-L_{i_h}(t_h-\tau-t_{h-1})} \right) \times \\ e_\tau^{-L_{i_j}(t_j-2\tau-s)}, & j \in \overline{q-2}, \\ e_\tau^{-L_{i_q}(t-t_{q-1})} e_\tau^{-L_{i_{q-1}}(t_{q-1}-2\tau-s)}, & j = q-1, \\ e_\tau^{-L_{i_q}(t-\tau-s)}, & j = q, \end{cases} \tag{9}$$

and

$$\phi_{q,j}(t,s) = \begin{cases} \sum_{M=1}^{q-j} \sum_{j \leqslant \xi_1 < \cdots < \xi_M \leqslant q-1} \int_{t_{\xi_M}-\tau}^{t_{\xi_M}} \cdots \int_{t_{\xi_1}-\tau}^{t_{\xi_1}} \\ \prod_{h=q}^{j} \zeta(h) ds_{\xi_1} \cdots ds_{\xi_M}, & j \in \overline{q-1}, \\ 0, & j = q, \end{cases} \tag{10}$$

where

$$\zeta(h) = \begin{cases} e_\tau^{-L_{i_q}(t-t_{q-1})}, & h = q, \ q-1 \notin \{\xi_k\}_{k=1}^M, \\ e_\tau^{-L_{i_q}(t-\tau-s_{q-1})}, & h = q, \ q-1 \in \{\xi_k\}_{k=1}^M, \\ e_\tau^{-L_{i_h}(t_h-\tau-t_{h-1})}, & j < h < q, \ h \notin \{\xi_k\}_{k=1}^M, \\ & h-1 \notin \{\xi_k\}_{k=1}^M, \\ e_\tau^{-L_{i_h}(t_h-2\tau-s_{h-1})}, & j < h < q, \ h \notin \{\xi_k\}_{k=1}^M, \\ & h-1 \in \{\xi_k\}_{k=1}^M, \\ -L_{i_h} e_\tau^{-L_{i_h}(s_h-\tau-t_{h-1})}, & j < h < q, h \in \{\xi_k\}_{k=1}^M, \\ & h-1 \notin \{\xi_k\}_{k=1}^M, \\ -L_{i_h} e_\tau^{-L_{i_h}(s_h-2\tau-s_{h-1})}, & j < h < q, h \in \{\xi_k\}_{k=1}^M, \\ & h-1 \in \{\xi_k\}_{k=1}^M, \\ e_\tau^{-L_{i_j}(t_j-2\tau-s)}, & h = j, \ j \notin \{\xi_k\}_{k=1}^M, \\ -L_{i_j} e_\tau^{-L_{i_j}(s_j-2\tau-s)}, & h = j, \ j \in \{\xi_k\}_{k=1}^M. \end{cases}$$

Further apply (9)-(10) to construct the following function:

$$\Phi_{q,j}(t,s)=\begin{cases} e_{\tau}^{-L_{i_1}(t-\tau-s)}, & s\in[t_0,t_1], \\ & j=1,\quad q=1, \\ \phi_{q,1}(t,s)+\psi_{q,1}(t,s), & s\in[0,t_1-\tau), \\ & j=1,\ q\in\overline{p}\setminus\{1\}, \\ \phi_{q,j}(t,s)+\psi_{q,j}(t,s), & s\in[t_{j-1}-\tau,t_j-\tau), \\ & 1<j<q,\ q\in\overline{p}\setminus\{1\}, \\ \phi_{q,q}(t,s)+\psi_{q,q}(t,s), & s\in[t_{q-1}-\tau,t_q], \\ & j=q,\ q\in\overline{p}\setminus\{1\}, \end{cases} \tag{11}$$

where $t\in[t_{q-1},t_q]$, $j\in\overline{q}$ and $q\in\overline{p}$. Then, we have the following lemma.

Lemma 1. *For any given switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, the explicit form of $z_q(\cdot)$, $q\in\overline{p}$, is represented as follows:*

$$z_q(t)=\begin{cases} \int_0^t \Phi_{1,1}(t,s)B_{i_1}u_{i_1}(s)ds, & q=1, \\ \sum_{j=1}^{q-1}\int_{t_{j-1}}^{t_j-\tau}\Phi_{q,j}(t,s)B_{i_j}u_{i_j}(s)ds \\ +\sum_{j=1}^{q-1}\int_{t_j-\tau}^{t_j}\Phi_{q,j+1}(t,s)B_{i_j}u_{i_j}(s)ds \\ +\int_{t_{q-1}}^{t}\Phi_{q,q}(t,s)B_{i_q}u_{i_q}(s)ds, q\in\overline{p}\setminus\{1\}, \end{cases} \tag{12}$$

where $\Phi_{q,j}(\cdot,\cdot),\ j\in\overline{q}$ is defined by (11).

Proof. Applying the properties of (1) and (9)-(11), we can directly calculate the result. □

B. Relative Controllability of the Switched System

In what follows, we consider the relative controllability of (5). We first present the corresponding definition as follows.

Definition 2. *The system* (5) *is called relatively controllable on $[0,t_p]$, if for any initial function $\varphi(t)$, $t\in[-\tau,0)$ and terminal state x_f, there exist a switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$ and a control input $u_{i_q}^*(\cdot)$, $q\in\overline{p}$ such that the solution of* (5) *satisfies $x_p^*(t_p)=x_f$ and $x_1^*(t)\equiv\varphi(t)$ for $t\in[-\tau,0)$.*

Theorem 1. *The system* (5) *is relatively controllable on $[0,t_p]$, if and only if there exists a switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, such that $W_c[0,t_p]$ is nonsingular, where*

$$\begin{aligned} W_c[0,t_p]\triangleq&\sum_{j=1}^{p-1}\int_{t_{j-1}}^{t_j-\tau}\Phi_{p,j}(t_p,s)B_{i_j}B_{i_j}^T\Phi_{p,j}^T(t_p,s)ds \\ &+\sum_{j=1}^{p-1}\int_{t_j-\tau}^{t_j}\Phi_{p,j+1}(t_p,s)B_{i_j}B_{i_j}^T\Phi_{p,j+1}^T(t_p,s)ds \\ &+\int_{t_{p-1}}^{t_p}\Phi_{p,p}(t_p,s)B_{i_p}B_{i_p}^T\Phi_{p,p}^T(t_p,s)ds. \end{aligned} \tag{13}$$

Proof. The proof is similar to that of [15] and is omitted here. □

Remark 1. *From* (13)*, we observe that the Gramian-type matrix depends closely on the time delay, the switching sequence and the matrix pairs $(-L_j,B_j)$ for $j\in\overline{r}$. For instance, altering the switching sequence leads to a corresponding change in the structure of $W_c[0,t_p]$. Therefore, compared with classical switched systems (e.g., [20]), the nonsingularity of $W_c[0,t_p]$ involves more factors. Moreover, due to the influence of delay, the choice of terminal time t_p also affects the nonsingularity of $W_c[0,t_p]$. As stated in Remark 2, if t_p falls below a certain critical value, $W_c[0,t_p]$ becomes singular. In fact, the Gramian matrix defined in* (13) *plays a fundamental role. It can span the set of controllable states (see Lemma 3 below), and thus helps to characterize the controllability property from a geometric perspective. Furthermore, as shown in Section V, it determines the minimum control energy required to achieve the control objective. If the MAS* (2)–(3) *is delay-free, then the controllability Gramian for $(-L_j,B_j)$, $j\in\overline{r}$, reduces to the form $W_c[t_0,t_p]=\sum_{j=1}^{p}\int_{t_{j-1}}^{t_j}\Omega(j,s)B_{i_j}B_{i_j}^T(\Omega(j,s))^Tds$ with $\Omega(j,s):=\prod_{k=p}^{j+1}e^{-L_{i_k}(t_k-t_{k-1})}e^{-L_{i_j}(t_j-s)}$.*

To further investigate the relative controllability of (5), we apply (12) to construct

$$\begin{aligned} \boldsymbol{\mathcal{Z}}\triangleq\Bigg\{z\,\Bigg|\,z=&\sum_{j=1}^{p-1}\int_{t_{j-1}}^{t_j-\tau}\Phi_{p,j}(t_p,s)B_{i_j}u_{i_j}^*(s)ds \\ &+\sum_{j=1}^{p-1}\int_{t_j-\tau}^{t_j}\Phi_{p,j+1}(t_p,s)B_{i_j}u_{i_j}^*(s)ds \\ &+\int_{t_{p-1}}^{t_p}\Phi_{p,p}(t_p,s)B_{i_p}u_{i_p}^*(s)ds,\ u_{i_j}^*\in\boldsymbol{\mathcal{U}},\ j\in\overline{p}\Bigg\}, \end{aligned}$$

where $\boldsymbol{\mathcal{U}}$ is the function space consisting of all measurable and square integrable vector functions.

Lemma 2. *The system* (5) *is relatively controllable on $[0,t_p]$, if and only if there exists a switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, such that $\boldsymbol{\mathcal{Z}}=\boldsymbol{\mathcal{R}}^n$.*

Proof. Apply (6) to construct that $\boldsymbol{\mathcal{X}}=\{x|x=y+z,z\in\boldsymbol{\mathcal{Z}},y$ is the solution of (7) for $q=p\}$. From Definition 2, we know that the relative controllability of (5) on $[0,t_p]$ is equivalent to $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{R}}^n$. If $\boldsymbol{\mathcal{X}}=\boldsymbol{\mathcal{R}}^n$, whereas $\boldsymbol{\mathcal{Z}}\subsetneq\boldsymbol{\mathcal{R}}^n$, then the solution of (7) satisfies $y\in\boldsymbol{\mathcal{Z}}^{\perp}$. This implies that the inner product $<y,z>=0$ for any $z\in\boldsymbol{\mathcal{Z}}$. From the structure of $y_q(\cdot)$, $q\in\overline{p}$, we know that y is a definite vector. Thus, the zero-forcing system (7) only possesses zero solution for any initial function. This is obviously impossible because L_j, $j\in\overline{r}$, being Laplacian matrices, are singular but nonzero. □

Lemma 3. *For any given switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, it holds that $\boldsymbol{\mathcal{Z}}=\mathbf{Col}(W_c[0,t_p])$.*

Proof. For any $\xi\in\mathbf{Col}(W_c[0,t_p])$, there exists a vector $\eta\in\boldsymbol{\mathcal{R}}^n$, such that $\xi=W_c[0,t_p]\eta$. Construct control inputs as follows:

$$u_{i_j}^*(t)=\begin{cases} B_{i_j}^T\Phi_{p,j}^T(t_p,t)\eta, & t\in[t_{j-1},t_j-\tau),\ j\in\overline{p-1}, \\ B_{i_j}^T\Phi_{p,j+1}^T(t_p,t)\eta, & t\in[t_j-\tau,t_j),\quad j\in\overline{p-1}, \\ B_{i_p}^T\Phi_{p,p}^T(t_p,t)\eta, & t\in[t_{p-1},t_p],\quad j=p. \end{cases} \tag{14}$$

Then, from (13) and (14), we have $\xi \in \mathcal{Z}$. This implies $\mathbf{Col}(W_c[0,t_p]) \subseteq \mathcal{Z}$.

For any nonzero $z \in \mathcal{Z}$, there exists $\widetilde{u}^*_{i_1}, \cdots, \widetilde{u}^*_{i_p} \in \mathcal{U}$, such that

$$\begin{aligned} z = &\sum_{j=1}^{p-1} \int_{t_{j-1}}^{t_j-\tau} \Phi_{p,j}(t_p,s) B_{i_j} \widetilde{u}^*_{i_j}(s) ds \\ &+ \sum_{j=1}^{p-1} \int_{t_j-\tau}^{t_j} \Phi_{p,j+1}(t_p,s) B_{i_j} \widetilde{u}^*_{i_j}(s) ds \\ &+ \int_{t_{p-1}}^{t_p} \Phi_{p,p}(t_p,s) B_{i_p} \widetilde{u}^*_{i_p}(s) ds. \end{aligned} \tag{15}$$

Assume that $z \notin \mathbf{Col}(W_c[0,t_p])$. Then we have $z \in \mathbf{Nul}(W_c[0,t_p])$. This implies

$$\begin{cases} z^T \Phi_{p,j}(t_p,t) B_{i_j} = 0, \ t \in [t_{j-1}, t_j - \tau), \ j \in \overline{p-1}, \\ z^T \Phi_{p,j+1}(t_p,t) B_{i_j} = 0, \ t \in [t_j - \tau, t_j), \ j \in \overline{p-1}, \\ z^T \Phi_{p,p}(t_p,t) B_{i_p} = 0, \qquad t \in [t_{p-1}, t_p]. \end{cases} \tag{16}$$

From (15)-(16), we have $z^T z = 0$, which contradicts $z \neq 0$. Thus, $\mathcal{Z} \subseteq \mathbf{Col}(W_c[0,t_p])$. We complete the proof. □

Corollary 1. *The system* (5) *is relatively controllable on* $[0, t_p]$, *if and only if there exists a switching sequence* $\mathcal{S} = \{(i_q, \kappa_q)\}_{q=1}^p$, *such that* $\mathbf{Col}(W_c[0,t_p]) = \mathcal{R}^n$.

Proof. From Lemmas 2-3, we obtain the result. □

For any given matrices $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$ and integer $\alpha \in \mathcal{N}^+$, define a subspace as follows:

$$\Gamma_A^\alpha \mathbf{Col} B = \mathbf{Col} B + A\mathbf{Col} B + \cdots + A^\alpha \mathbf{Col} B. \tag{17}$$

If $\alpha \geqslant n-1$, according to Hamilton-Cayley theorem, we know that $\Gamma_A^\alpha \mathbf{Col} B$ is equivalent to $\langle A|B\rangle$ defined in [19]. If $\alpha < n-1$, we have $\Gamma_A^\alpha \mathbf{Col} B \subseteq \langle A|B\rangle$. Thus, for any $\alpha \in \mathcal{N}^+$, it always holds that $\Gamma_A^\alpha \mathbf{Col} B \subseteq \langle A|B\rangle$. For the matrices $A_1, A_2 \in \mathcal{R}^{n\times n}$ and positive integers α_1, α_2, we define a nested subspace as follows:

$$\begin{aligned} &\Gamma_{A_2}^{\alpha_2} \Gamma_{A_1}^{\alpha_1} \mathbf{Col} B = \\ &\Gamma_{A_1}^{\alpha_1} \mathbf{Col} B + A_2 \Gamma_{A_1}^{\alpha_1} \mathbf{Col} B + \cdots + A_2^{\alpha_2} \Gamma_{A_1}^{\alpha_1} \mathbf{Col} B. \end{aligned} \tag{18}$$

Similarly, when $\alpha_1 \geqslant n-1, \ \alpha_2 \geqslant n-1$, based on Hamilton-Cayley theorem, the subspace (18) equals $\langle A_2|\mathcal{W}\rangle$ defined in [19], where $\mathcal{W} = \langle A_1|B\rangle$. Conversely, if one of α_1 and α_2 is less than $n-1$, then we have $\Gamma_{A_2}^{\alpha_2} \Gamma_{A_1}^{\alpha_1} \mathbf{Col} B \subseteq \langle A_2|\mathcal{W}\rangle$. Thus, for any $\alpha_1, \alpha_2 \in \mathcal{N}^+$, we always have $\Gamma_{A_2}^{\alpha_2} \Gamma_{A_1}^{\alpha_1} \mathbf{Col} B \subseteq \langle A_2|\mathcal{W}\rangle$.

Apply (17)-(18) to construct a nested subspace $\mathcal{V}_{q,j}, j \in \overline{q}$, $q \in \overline{p}$ along the switching sequence $\mathcal{S} = \{(i_q, \kappa_q)\}_{q=1}^p$ as follows:

$$\mathcal{V}_{q,j} = \begin{cases} \Gamma_{L_{i_q}}^{l_q} \cdots \Gamma_{L_{i_{j+1}}}^{l_{j+1}} \Gamma_{L_{i_j}}^{l_j - 1} \mathbf{Col} B_{i_j}, \ j \in \overline{q-1}, \\ \Gamma_{L_{i_q}}^{l_q - 1} \mathbf{Col} B_{i_q}, \qquad j = q, \end{cases} \tag{19}$$

Then, we have the following lemma:

Lemma 4. *For any given switching sequence* $\mathcal{S} = \{(i_q, \kappa_q)\}_{q=1}^p$, *it holds that* $z_q(t_q) \in \sum_{j=1}^q \mathcal{V}_{q,j}, \ q \in \overline{p}$.

Proof. For $q = 1$, it follows from [17] that

$$z_1(t_1) \in \Gamma_{L_{i_1}}^{l_1 - 1} \mathbf{Col} B_{i_1} = \mathcal{V}_{1,1}.$$

For $q = 2$, from (9)-(11) and Lemma 1, we have

$$\begin{aligned} z_2(t_2) = &\int_0^{t_1-\tau} e_\tau^{-L_{i_2}(t_2-t_1)} e_\tau^{-L_{i_1}(t_1-2\tau-s)} B_{i_1} u^*_{i_1}(s) ds \\ &- \int_0^{t_1-\tau} \int_{t_1-\tau}^{t_1} e_\tau^{-L_{i_2}(t_2-\tau-s_1)} L_{i_1} \\ &\times e_\tau^{-L_{i_1}(s_1-2\tau-s)} B_{i_1} u^*_{i_1}(s) ds_1 ds \\ &+ \int_{t_1-\tau}^{t_1} e_\tau^{-L_{i_2}(t_2-\tau-s)} B_{i_1} u^*_{i_1}(s) ds \\ &+ \int_{t_1}^{t_2} e_\tau^{-L_{i_2}(t_2-\tau-s)} B_{i_2} u^*_{i_2}(s) ds. \end{aligned} \tag{20}$$

According to the properties of matrix delayed exponential (1), we have

$$z_2(t_2) = \sum_{j_2=0}^{l_2} L_{i_2}^{j_2} \sum_{j_1=0}^{l_1-1} L_{i_1}^{j_1} B_{i_1} f_1(j_2, j_1) + \sum_{j_2=0}^{l_2-1} L_{i_2}^{j_2} B_{i_2} f_2(j_2), \tag{21}$$

where $f_1(j_2, j_1) =$

$$\begin{cases} \int_{-\tau}^{(l_1-2)\tau} (-1)^{j_2} \frac{(t_2-(j_2-1)\tau)^{j_2}}{j_2!} u^*_{i_1}(t_2-\tau-\eta) d\eta + \\ \int_{(l_2-1)\tau}^{l_2\tau} (-1)^{j_2} \frac{(\eta-(j_2-1)\tau-t_1)^{j_2}}{j_2!} \times \\ u^*_{i_1}(t_1-2\tau-\eta) d\eta, \qquad j_1 = 0, \\ \int_{(l_2-1)\tau}^{l_2\tau} \int_0^{t_2-(j_1+1)\tau-\eta} (-1)^{j_2+j_1} \frac{(\eta-(j_2-1)\tau)^{j_2}}{j_2!} \times \\ \frac{(t_2-(j_1+1)\tau-\eta-s)^{j_1-1}}{(j_1-1)!} u^*_{i_1}(s) ds d\eta + \\ \int_{(j_1-1)\tau}^{(l_1-2)\tau} (-1)^{j_2+j_1} \frac{(t_2-(j_2-1)-t_1)^{j_2} (\eta-(j_1-1)\tau)^{j_1}}{j_2! j_1!} \times \\ u^*_{i_1}(t_1-2\tau-\eta) d\eta, \qquad j_1 \in \overline{l_1-2}, \\ \int_{(l_2-1)\tau}^{l_2\tau} \int_0^{t_2-l_1\tau-\eta} (-1)^{j_2+l_1-1} \frac{(\eta-(j_2-1)\tau)^{j_2}}{j_2!} \times \\ \frac{(\eta-(j_2-1)\tau)^{j_2}}{(l_1-2)!} u^*_{i_1}(s) ds d\eta, \qquad j_1 = l_1 - 1, \end{cases}$$

and $f_2(j_2) =$

$$\int_{(j_2-1)\tau}^{(l_2-1)\tau} (-1)^{j_2} \frac{(\eta-(j_2-1)\tau)^{j_2}}{j_2!} B_{i_2} u^*_{i_2}(t_2-\tau-\eta) d\eta.$$

Thus from (19) and (21), it yields that

$$z_2(t_2) \in \Gamma_{L_{i_2}}^{l_2} \Gamma_{L_{i_1}}^{l_1-1} \mathbf{Col} B_{i_1} + \Gamma_{L_{i_2}}^{l_2-1} \mathbf{Col} B_{i_2} = \sum_{j=1}^2 \mathcal{V}_{2,j}.$$

For $q = h, \ h \in \overline{q} \setminus \{1, 2\}$, following similar processes, we have

$$\begin{aligned} z_h(t_h) = &\sum_{\xi=1}^{h-1} \left(\prod_{g=h}^{\xi+1} \sum_{j_g=0}^{l_g} L_{i_g}^{j_g} \right) \sum_{j_\xi=0}^{l_\xi-1} L_{i_\xi}^{j_\xi} B_{i_\xi} f_\xi(j_h, \cdots, j_\xi) \\ &+ \sum_{j_h=0}^{l_h-1} L_{i_h}^{j_h} B_{i_h} f_h(j_h) \end{aligned} \tag{22}$$

where $f_1, \cdots, f_h$ are linear functionals of inputs $u^*_{i_1}, \cdots, u^*_{i_h}$, respectively. Thus, from (19), we obtain $z_h(t_h) \in \sum_{j=1}^h \mathcal{V}_{h,j}$. The proof is completed. □

Applying above conclusions we have below assertion.

Lemma 5. *For any given switching sequence* $\mathcal{S} = \{(i_q,\kappa_q)\}_{q=1}^{p}$, *we have* $\sum_{j=1}^{p}\boldsymbol{\mathcal{V}}_{p,j} = \mathbf{Col}(W_c[0,t_p])$.

Proof. For any $\gamma \in \mathbf{Nul}\,(W_c[0,t_p])$, from (16), we have

$$\begin{cases}\gamma^T\Phi_{p,j}(t_p,t+d\tau)L_{i_j}^d B_{i_j}{=}0, & t\in[t_{j-1},t_j-\tau),\\ & d\in\overline{l_j-2}\cup\{0\},\ j\in\overline{p-1},\\ \gamma^T\Phi_{p,p}(t_p,t+d\tau)L_{i_p}^d B_{i_p}{=}0, t{\in}[t_{p-1},t_p], d{\in}\overline{l_p-1}\cup\{0\}.\end{cases} \tag{23}$$

Taking $t=t_j-(d+2)\tau,\ j\in\overline{p-1}$ and $t=t_p-(d+1)\tau$ in (23), respectively, we further obtain

$$\begin{cases}\gamma^T\Phi_{p,j}(t_p,t_j-2\tau)L_{i_j}^d B_{i_j}=0,\ d\in\overline{l_j-2}\cup\{0\}, j\in\overline{p-1},\\ \gamma^T\Phi_{p,p}(t_p,t_p-\tau)L_{i_p}^d B_{i_p}=0, \qquad d\in\overline{l_p-1}\cup\{0\}.\end{cases}$$

Thus, we have

$$\begin{aligned}\gamma \in &\left(\bigcap_{j=1}^{p-1}\bigcap_{d=0}^{l_j-2}\mathbf{Nul}\left(B_{i_j}^T(L_{i_j}^d)^T\Phi_{p,j}^T(t_p,t_j-2\tau)\right)\right)\\ &\bigcap\left(\bigcap_{d=0}^{l_p-1}\mathbf{Nul}\left(B_{i_p}^T(L_{i_p}^d)^T\Phi_{p,p}^T(t_p,t_p-\tau)\right)\right)\\ =&\left(\bigcap_{j=1}^{p-1}\bigcap_{d=0}^{l_j-2}\left(\mathbf{Col}\left(\Phi_{p,j}(t_p,t_j-2\tau)L_{i_j}^d B_{i_j}\right)\right)^{\perp}\right)\\ &\bigcap\left(\bigcap_{d=0}^{l_p-1}\left(\mathbf{Col}\left(\Phi_{p,p}(t_p,t_p-\tau)L_{i_p}^d B_{i_p}\right)\right)^{\perp}\right)\\ =&\left(\sum_{j=1}^{p-1}\sum_{d=0}^{l_j-2}\mathbf{Col}\left(\Phi_{p,j}(t_p,t_j-2\tau)L_{i_j}^d B_{i_j}\right)\right.\\ &\left.+\sum_{d=0}^{l_p-1}\mathbf{Col}\left(\Phi_{p,p}(t_p,t_p-\tau)L_{i_p}^d B_{i_p}\right)\right)^{\perp}.\end{aligned}$$

For the matrices L_j and B_j in (4), $j\in\overline{r}$, applying the properties $L_j^T=L_j$ and $L_j\mathbf{1}_n=\mathbf{0}$, we can prove that $(\prod_{k=1}^m L_{i_k}+\prod_{k=1}^m L_{j_k})\mathbf{Col}B_l=\prod_{k=1}^m L_{i_k}\mathbf{Col}B_l+\prod_{k=1}^m L_{j_k}\mathbf{Col}B_l$ for any $m,l\in\overline{p}$ and $i_k,j_k\in\overline{r}$, where $k\in\overline{m}$. It thus follows from (1), (11), and (19) that $\gamma\in\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j}$. This implies $\mathbf{Nul}\,(W_c[0,t_p])\subseteq(\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j})^{\perp}$. From Lemmas 3 and 4, we immediately obtain $\mathbf{Col}\,(W_c[0,t_p])\subseteq\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j}$. This completes the proof. □

Theorem 2. *The system* (5) *is relatively controllable on* $[0,t_p]$, *if and only if there exists a switching sequence* $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^p$, *such that* $\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j}=\boldsymbol{\mathcal{R}}^n$.

Proof. From Lemmas 2, 3 and 5, we obtain the result directly. This completes the proof. □

Corollary 2. *If* $\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j}=\boldsymbol{\mathcal{R}}^n$, *then we have* $t_p\geqslant t_p^*$, *where* t_p^* *is the critical terminal time defined by*

$$\begin{aligned}t_p^*=&\left\lceil\frac{n+m(i_p)}{\sum_{\xi=1}^{p-2}\left(\left(\prod_{g=p-1}^{\xi+1}(l_g+1)\right)l_\xi m(i_\xi)\right)+l_{p-1}m(i_{p-1})+m(i_p)}\right\rceil\tau\\ &+\sum_{\xi=1}^{p-1}l_\xi\tau-\tau.\end{aligned} \tag{24}$$

Proof. From Theorem 2, we know that the system (5) is relatively controllable on $[0,t_p]$. This implies that for any initial function $\varphi(t),\ \ t\in[-\tau,0)$ and terminal state x_f, $x_f=y_p^*(t_p)+z_p(t_p)$ always has a solution with respect to $u_{i_1}^*,\cdots,u_{i_p}^*$, where $y_p^*(t_p)$ is the solution of (7) and $z_p(t_p)$ is (12). This is equivalent to that (22) for $h=p$ has a solution with respect to $f_1,\cdots,f_p$. If $\sum_{\xi=1}^{p-1}((\prod_{g=p}^{\xi+1}(l_g+1))l_\xi m(i_\xi))+l_p m(i_p)<n$, then $x_f-y_p^*(t_p)=z_p(t_p)$ might not admit such a solution. Thus, we obtain the result. □

Remark 2. *If the communication topology of the MAS is fixed, then* (5) *is reduced to the well-known delay system* $(-L,B)_\tau$ *and the running interval is* $[t_0,t_f]$, *where* $L\in\boldsymbol{\mathcal{R}}^{n\times n}$, $B\in\boldsymbol{\mathcal{R}}^{n\times m}$, *and* t_f *is the terminal time for the topology-fixed case satisfying* $t_f=k\tau,\ k\in\boldsymbol{\mathcal{N}}^+$. *Meanwhile, the condition in Theorem 2 reduces to* $\Gamma_L^{k-1}\mathbf{Col}B=\boldsymbol{\mathcal{R}}^n$, *where* $k\in\boldsymbol{\mathcal{N}}^+$ *is the integer such that* $k\tau=t_f$. *From [3], we know that the relative controllability of system* $(-L,B)_\tau$ *on* $[0,t_f]$ *is characterized by* $\Gamma_L^{n-1}\mathbf{Col}B=\boldsymbol{\mathcal{R}}^n$ *and* $t_f\geqslant t_f^*:=\lceil n/m\rceil\tau$. *However, for the relative controllability of* (5), *we remove the explicit condition on terminal time. This is owing to the subspaces* (17) *and* (18) *that we construct. In fact, from Corollary 2, we know that if* $t_p<t_p^*$, *then* $\sum_{j=1}^p\boldsymbol{\mathcal{V}}_{p,j}$ *does not equal the whole space* $\boldsymbol{\mathcal{R}}^n$. *This implies that the system* (5) *is not relatively controllable on* $[0,t_p]$. *In comparison, if the system* (5) *is not affected by the time delay, the controllability of the switched system* $(-L_j,B_j),\ j\in\overline{r}$ *has nothing to do with the terminal time [20].*

Next, we continue to discuss the relative controllability criterion of the system (5). Our goal is to establish a Kalman-type matrix criterion on relative controllability of the system.

For any given matrices $A\in\boldsymbol{\mathcal{R}}^{n\times n}$, $B\in\boldsymbol{\mathcal{R}}^{n\times m}$ and integer $\alpha\in\boldsymbol{\mathcal{N}}^+$, define a block matrix as follows:

$$\Lambda_A^\alpha B=[B,AB,\cdots,A^\alpha B]\,. \tag{25}$$

Notice that if $\alpha=n-1$, then $\Lambda_A^\alpha B$ is the Kalman-type matrix corresponding to the system (A,B). Furthermore, the space spanned by the columns of $\Lambda_A^\alpha B$ is (17). For the matrices $A_1,A_2\in\boldsymbol{\mathcal{R}}^{n\times n}$ and positive integers α_1,α_2, further construct a block matrix as follows:

$$\Lambda_{A_2}^{\alpha_2}\Lambda_{A_1}^{\alpha_1}B=\left[\Lambda_{A_1}^{\alpha_1}B,A_2\Lambda_{A_1}^{\alpha_1}B,\cdots,A_2^{\alpha_2}\Lambda_{A_1}^{\alpha_1}B\right]. \tag{26}$$

It is obvious that $\mathbf{Col}\Lambda_{A_2}^{\alpha_2}\Lambda_{A_1}^{\alpha_1}B=\Gamma_{A_2}^{\alpha_2}\Gamma_{A_1}^{\alpha_1}\mathbf{Col}B$.

For any given switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, we further apply (25)-(26) to construct a block matrix $\widetilde{Q}_j,\ j\in\overline{p}$ as follows:

$$\widetilde{Q}_j=\begin{cases}\Lambda_{L_{i_p}}^{l_p}\cdots\Lambda_{L_{i_{j+1}}}^{l_{j+1}}\Lambda_{L_{i_j}}^{l_j-1}B_{i_j}, & j\in\overline{p-1},\\ \Lambda_{L_{i_p}}^{l_p-1}B_{i_p}, & j=p,\end{cases}\tag{27}$$

where $i_j\in\overline{r}$, $j\in\overline{p}$. Then for the relative controllability of (5), we have the following conclusion.

Theorem 3. *The system* (5) *is relatively controllable on* $[0,t_p]$, *if and only if there exists a switching sequence* $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, *such that the controllability matrix* Q_c *is full row rank, where* $Q_c=[\widetilde{Q}_1,\widetilde{Q}_2,\cdots,\widetilde{Q}_p]$.

Proof. From (25) and (26), it holds that $\mathbf{Col}\Lambda_{L_{i_p}}^{l_p-1}B_{i_p}=\Gamma_{L_{i_p}}^{l_p-1}\mathbf{Col}B$ for $j=p$ and that $\mathbf{Col}\Lambda_{L_{i_p}}^{l_p}\Lambda_{L_{i_{p-1}}}^{l_{p-1}-1}B_{i_{p-1}}=\Gamma_{L_{i_p}}^{l_p}\Gamma_{L_{i_{p-1}}}^{l_{p-1}-1}\mathbf{Col}B_{i_{p-1}}$ for $j=p-1$. By applying the backward induction, we obtain that $\mathbf{Col}\Lambda_{L_{i_p}}^{l_p}\cdots\Lambda_{L_{i_{j+1}}}^{l_{j+1}}\Lambda_{L_{i_j}}^{l_j-1}B_{i_j}=\Gamma_{L_{i_p}}^{l_p}\cdots\Gamma_{L_{i_{j+1}}}^{l_{j+1}}\Gamma_{L_{i_j}}^{l_j-1}\mathbf{Col}B_{i_j}$ for $j\in\overline{p-2}$. Thus, from (27) and the definitions of Q_c, we have $\mathbf{Col}Q_c=\mathbf{Col}\widetilde{Q}_1+\mathbf{Col}\widetilde{Q}_2+\cdots+\mathbf{Col}\widetilde{Q}_p=\Gamma_{L_{i_p}}^{l_p-1}\mathbf{Col}B_{i_p}+\Gamma_{L_{i_p}}^{l_p}\Gamma_{L_{i_{p-1}}}^{l_{p-1}-1}\mathbf{Col}B_{i_{p-1}}+\cdots+\Gamma_{L_{i_p}}^{l_p}\cdots\Gamma_{L_{i_2}}^{l_2}\Gamma_{L_{i_1}}^{l_1-1}\mathbf{Col}B_{i_1}$. It thus follows from (19) that $\mathbf{Col}Q_c=\sum_{j=1}^{p}\boldsymbol{\mathcal{V}}_{p,j}$. Combined with Theorem 2, we conclude that the relative controllability of the system (5) on $[0,t_p]$ is equivalent to $\mathbf{Col}Q_c=\boldsymbol{\mathcal{R}}^n$, which can be further explained as Q_c being full row rank. This completes the proof. □

Remark 3. *From* (27)*, we know that* Q_c *is closely related to the terminal time* t_p*. In fact, from Corollary 2 and Theorem 3, we know that if the terminal time* $t_p<t_p^*$*, then* Q_c *is not full row rank. Consequently, the system* (5) *is not relatively controllable on* $[0,t_p]$ *under any switching sequence. Meanwhile, if the system* (5) *is reduced to the delay system* $(-L,B)_\tau$ *in Remark 2, then it follows from [3] that* Q_c *is reduced to* $[B,LB,\cdots,L^{k-1}B]$*. Thus, the relative controllability criterion for system* $(-L,B)_\tau$ *on* $[0,t_f]$ *reduces to verifying the full row rank of* $[B,LB,\cdots,L^{k-1}B]$.

V. The Minimum Energy for Relative Controllability

Next, we characterize the external control effort required to achieve relative controllability of the MAS. This problem is formulated as the minimization of the following functional index with respect to the control functions over a finite time horizon:

$$J(u_{i_1}^*,\cdots,u_{i_p}^*)=\sum_{j=1}^{p}\int_{t_{j-1}}^{t_j}u_{i_j}^{*T}(s)Q_ju_{i_j}^*(s)ds.\tag{28}$$

Construct a matrix as follows:

$$\begin{aligned}\widehat{W}\ =\ &\sum_{j=1}^{p-1}\int_{t_{j-1}}^{t_j-\tau}\Phi_{p,j}(t_p,s)B_{i_j}Q_j^{-1}B_{i_j}^T\Phi_{p,j}^T(t_p,s)ds\\&+\sum_{j=1}^{p-1}\int_{t_j-\tau}^{t_j}\Phi_{p,j+1}(t_p,s)B_{i_j}Q_j^{-1}B_{i_j}^T\Phi_{p,j+1}^T(t_p,s)ds\\&+\int_{t_{p-1}}^{t_p}\Phi_{p,p}(t_p,s)B_{i_p}Q_p^{-1}B_{i_p}^T\Phi_{p,p}^T(t_p,s)ds,\end{aligned}\tag{29}$$

where $Q_j,\ j\in\overline{p}$ are constant symmetric and positive definite weighting matrices. Note that the dimensions of matrices $Q_j,\ j\in\overline{p}$ are varying with switching function σ. Following a similar process to Theorem 1, we conclude that the relative controllability of (5) on $[0,t_p]$ is equivalent to verifying the nonsingularity of the matrix $\widehat{W}$ for some switching sequence $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$. With $\widehat{W}$ being nonsingular, construct that

$$\widehat{u}_{i_j}^*(t)=\begin{cases}Q_j^{-1}B_{i_j}^T\Phi_{p,j}^T(t_p,t)\widehat{W}^{-1}\left(x_f-y_p^*(t_p)\right),\\ \qquad t\in[t_{j-1},t_j-\tau),\ j\in\overline{p-1},\\ Q_j^{-1}B_{i_j}^T\Phi_{p,j+1}^T(t_p,t)\widehat{W}^{-1}\left(x_f-y_p^*(t_p)\right),\\ \qquad t\in[t_j-\tau,t_j),\quad j\in\overline{p-1},\\ Q_p^{-1}B_{i_p}^T\Phi_{p,p}^T(t_p,t)\widehat{W}^{-1}\left(x_f-y_p^*(t_p)\right),\\ \qquad t\in[t_{p-1},t_p],\qquad j=p.\end{cases}\tag{30}$$

The symbols in (30) are the same as those in Section IV. Then we conclude that (30) can minimize the functional index (28).

Theorem 4. *Given a switching sequence* $\mathcal{S}=\{(i_q,\kappa_q)\}_{q=1}^{p}$, *assume that* (29) *is nonsingular. Then* (30) *minimizes the functional index* (28)*. The minimal value of* J *is*

$$J(\widehat{u}_{i_1}^*,\cdots,\widehat{u}_{i_p}^*)=z_p^T(t_p)\widehat{W}^{-1}z_p(t_p).\tag{31}$$

Proof. Following the assumption, we know that for any initial function $\varphi(t),\ t\in[-\tau,0)$ and terminal state $x_f\in\boldsymbol{\mathcal{R}}^n$, (30) can steer $\varphi(t)$ to x_f. If $\widetilde{u}_{i_j}^*(\cdot),\ j\in\overline{p}$ is another function which can steer $\varphi(t)$ to x_f. Then, from (6), Lemma 1, and the solution of (7), we have

$$\begin{aligned}0=&\sum_{j=1}^{p-1}\int_{t_{j-1}}^{t_j-\tau}\Phi_{p,j}(t_p,s)B_{i_j}\left(\widetilde{u}_{i_j}^*(s)-\widehat{u}_{i_j}^*(s)\right)ds\\&+\sum_{j=1}^{p-1}\int_{t_j-\tau}^{t_j}\Phi_{p,j+1}(t_p,s)B_{i_j}\left(\widetilde{u}_{i_j}^*(s)-\widehat{u}_{i_j}^*(s)\right)ds\\&+\int_{t_{p-1}}^{t_p}\Phi_{p,p}(t_p,s)B_{i_p}\left(\widetilde{u}_{i_p}^*(s)-\widehat{u}_{i_p}^*(s)\right)ds.\end{aligned}\tag{32}$$

Combining with (29)-(32), we obtain

$$J(\widehat{u}_{i_1}^*,\cdots,\widehat{u}_{i_p}^*)=\sum_{j=1}^{p}\int_{t_{j-1}}^{t_j}\widetilde{u}_{i_j}^{*T}(s)Q_j\widehat{u}_{i_j}^*(s)ds.\tag{33}$$

It thus follows that $J(\widetilde{u}_{i_1}^*,\cdots,\widetilde{u}_{i_p}^*)-J(\widehat{u}_{i_1}^*,\cdots,\widehat{u}_{i_p}^*)=\sum_{j=1}^{p}\int_{t_{j-1}}^{t_j}\left(\widetilde{u}_{i_j}^*(s)-\widehat{u}_{i_j}^*(s)\right)^TQ_j\left(\widetilde{u}_{i_j}^*(s)-\widehat{u}_{i_j}^*(s)\right)ds>0$. Therefore, we obtain the following conclusion: $J(\widehat{u}_{i_1}^*,\cdots,\widehat{u}_{i_p}^*)<J(\widetilde{u}_{i_1}^*,\cdots,\widetilde{u}_{i_p}^*)$. Applying (28)-(30), we obtain the minimum energy as follows:

$J(\widehat{u}_{i_1}^*, \cdots, \widehat{u}_{i_p}^*) = \sum_{j=1}^{p} \int_{t_{j-1}}^{t_j} \widehat{u}_{i_j}^{*T}(s) Q_j \widehat{u}_{i_j}^*(s) ds = (x_f - y_p^*(t_p))^T \widehat{W}^{-1} (x_f - y_p^*(t_p))$. The conclusion is immediately obtained from (6). □

Remark 4. *The structure of* $\widehat{W}$ *in* (29) *is related to the switching sequence. This implies that* (31) *varies with the switching sequence. Namely, the minimal energy* $J(\widehat{u}_{i_1}^*, \cdots, \widehat{u}_{i_p}^*)$ *in* (31) *is a local property. To achieve the global minimum energy, we need to compare the minimal energy values corresponding to all switching sequences that render the system relatively controllable. This discloses the complexity of the switched systems with delay compared with the literature [21]. In fact, if we let* $\boldsymbol{\Delta}$ *be the set of all switching sequences that ensure* $\widehat{W}$ *is nonsingular, then the minimum energy achieving control objective is* $\min_{\mathcal{S} \in \boldsymbol{\Delta}} J(\widehat{u}_{i_1}^*, \cdots, \widehat{u}_{i_p}^*)(\mathcal{S}) = \min_{\mathcal{S} \in \boldsymbol{\Delta}} z_p^T(t_p) \widehat{W}^{-1} z_p(t_p)(\mathcal{S})$. *Note that the minimum energy is characterized by the inverse of* $\widehat{W}$. *Thus, when* $\widehat{W}$ *is singular, there exists no finite energy that can steer the system to reach any component of the target state that lies in the null space of* $\widehat{W}$, *and it thus fails to characterize the minimum control effort required for relative controllability of the MAS.*

VI. Example

Finally, we provide an example to illustrate our work. Consider a multi-vehicle system consisting of eight vehicles with communication delay and three switching topologies (each topology is abstracted as in Fig. 1). Assume that the switching sequence of the MAS is $\mathcal{S} = \{(1,1),(2,1),(3,1)\}$ and the time delay is $\tau = 0.1$. For simplicity, we assign the communication weight of the edge (i,j) as $w_{ji} = 1$ in each topology. Then from (4) and Fig. 1, we can easily obtain the Laplacian matrices L_1, L_2, L_3 and the input matrices B_1, B_2, B_3. Other parameters are chosen as $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$ and $l_1 = l_2 = l_3 = 10$. From (19), we have $\sum_{j=1}^{3} \boldsymbol{\mathcal{V}}_{3,j} =$ **Span** $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} = \boldsymbol{\mathcal{R}}^8$, where e_j for $j \in \overline{8}$ are the standard basis vectors. From Theorem 2, we conclude that the MAS is relatively controllable on $[0, 3]$. Simple numerical calculation shows that the critical terminal time is $t_3^* = 2 < 3$ and the matrix $W_c[0,3]$ in Theorem 1 is nonsingular. This is consistent with our theoretical findings.

If we set $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.4$, while keeping the other conditions unchanged, then a straightforward calculation gives $\sum_{j=1}^{3} \mathcal{V}_{3,j} = \mathcal{R}^8$; that is, the MAS is relatively controllable on $[0, 0.4]$. Note that the critical terminal time is $t_3 = t_3^* = 0.4$ in this case. However, if we change the terminal time to $t_3 = 0.3$ while keeping the other conditions fixed, then we have $t_3^* = 0.4 > 0.3 = t_3$. From Corollary 2, we conclude that the MAS is **not** relatively controllable on $[0, 0.3]$. Due to the space limitation, here we omit the discussion on other controllability criteria and the minimum energy problem.

VII. Conclusion

This paper considers the relative controllability of MASs subject to communication delay and switching topologies. A new subspace is constructed, and based on this, a controllable subspace criterion is established. Additionally, a novel Kalman-type matrix criterion is derived, generating the conclusion for classical delay systems. It is shown that if the terminal time is less than some critical value, then the MAS is not relatively controllable due to the influence of delay. To characterize the effort of external control in steering the MAS towards its terminal goal, the problem is formulated as minimizing the functional index defined over the control function. An optimal control and the corresponding minimum value of the index are given.

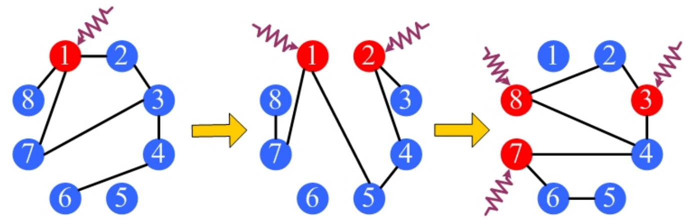

Fig. 1: Switching topologies of the MAS (2)-(3).

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